Existence of Solitary Internal Waves in a Two-Layer Fluid of Infinite Height

S. M. Sun

Department of Mathematics

Virginia Polytechnic Institute

and State University

Blacksburg, VA 24061

Abstract

This paper concerns the existence of internal solitary waves moving with a constant speed at the interface of a two-layer fluid with infinite height. The fluids are immiscible, inviscid, and incompressible with constant but different densities. Assume that the height of the upper fluid is infinite and the depth of the lower fluid is finite. It has been formally derived before that under long-wave assumption the first-order approximation of the interface satisfies the Benjamin-Ono equation, which has algebraic solitary-wave solutions. This paper gives a rigorous proof of the existence of solitary-wave solutions of the exact equations governing the fluid motion, whose first-order approximations are the algebraic solitary-wave solutions of the Benjamin-Ono equation. The proof relies on estimates of integral operators using Fourier transforms in $L^2(\mathbf{R})$-space and is different from the previous existence proof of solitary waves in a two-layer fluid with finite depth.

1. Introduction

This paper deals with internal waves of finite amplitude in a stable two-fluid system with infinite height. The two fluids are immiscible, inviscid, and incompressible with different but constant densities. The lower boundary is horizontal and no upper boundary exists. Let the depth of the lower fluid be h in an equilibrium state and the densities of the lower and upper layers of the flow be ρ^-, ρ^+, respectively, with $\rho^- > \rho^+ > 0$. It is of interest to find an internal solitary wave moving with a constant velocity U at the interface. This problem was first studied by Benjamin using long-wave theory [4]. In the long-wave analysis, the lengths of waves at the interface are assumed to be considerably larger than h, but no long-wave approximation is possible for the motion of the upper fluid since the height of the upper fluid is infinite. In reference to an (x^*, y^*)-coordinate system moving with the same speed U, the fluid motion is steady. It is found in [4] that to have long waves at the interface, U must be near a critical wave speed $U_0 = ((\rho^- - \rho^+)gh/\rho^-)^{1/2}$, where g is the gravitational acceleration constant. Assume that $(U_0/U)^2 = 1 + \gamma_1\epsilon$ with $\epsilon > 0$ a small long-wave parameter and $\gamma_1 < 0$ a fixed constant and the long-wave expansion of the solution in the lower fluid is used. A model equation, called Benjamin-Ono equation, can be derived as follows,

$$\frac{\rho^+ h}{\rho^-}\mathcal{F}(\eta^*) - \gamma_1\eta^* - \left(\frac{3\eta^{*2}}{2h}\right) = 0\,, \tag{1}$$

where

$$\mathcal{F}(\eta^*) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx}|k|\int_{-\infty}^{+\infty} e^{ik\xi}\eta^*(\xi)d\xi dk\,,$$

and $\eta^*(x)$ is the deviation of the interface from the equilibrium state. The equation (1) has a solitary-wave solution

$$S(x) = \frac{a\mu^2h^3}{x^2 + (h\mu)^2}\,, \tag{2}$$

where $a = -(4\gamma_1/3), \mu = -(\rho^+/\rho^-\gamma_1)$ and x is a stretched horizontal variable with $x = \epsilon x^*$.

The Benjamin-Ono equation was first derived in [4] for very general stratified fluids with infinite depth and the two-fluid flow is one of these cases. Also the algebraic solitary-wave solution (2) was obtained using Fourier transform technique. At the same time, Davis and Acrivos [11] obtained the equation in an experimental and numerical study of the wave

phenomena. Then Ono [19] derived a corresponding time-dependent equation, which was later studied using inverse scattering method [9, 12] and has multi-soliton solutions [17]. Maxworthy [18] carried out experiments again to show the existence of internal solitary waves and found that the waves are quite stable. Maslowe and Redekopp [16] generalized the Benjamin-Ono equation to nonlinear waves in stratified shear flows and obtained more complicated integral model equations. Then Benjamin [6] considered a two-layer fluid with large surface tension at interface and derived the equation (1) with an extra term of third-order derivative. Recently Amick and Toland [2, 3] proved that the solitary-wave solution (2) and its translations in the x-direction are the only solutions of (1) which decay to zero as x approaches infinity. Bennett et al. [8] obtained the stability of solitary waves of the Benjamin-Ono equation with time dependence by a similar technique used in the proof of stability of solitary waves for the KdV equation [5, 10]. However, we note that the solitary-wave solution (2) is a solution of the Benjamin-Ono equation, which is an approximate equation of the exact equations governing the fluid motion. It is importment to know whether the solitary-wave solution (2) is an approximation of a solution of the exact equations. The corresponding problem for two-fluid flows with finite depth has been studied intensively in last fifteen years. The model equation for the interface is so-call KdV equation, which also has solitary-wave solutions. It has been proved by numerous literatures that the solitary-wave solution of the KdV equation is an approximation of a solitary-wave solution of the exact equations. The reader is referred to an article by Benjamin et al. [7] for a general treatment of such problems and the references therein. As pointed out in [7], the question whether the solitary-wave solution (2) is an approximation of a solution of the exact equations for a two-fluid flow with infinite depth needs to be answered.

This paper gives a rigorous proof of the existence of solitary waves at the interface for a two-layer fluid with infinite depth, whose first order approximations are the solitary waves given by (2). The result can be stated as follows. Assume that the depth of lower fluid is h at infinity and the height of the upper fluid is infinite. The densities are ρ^+ and ρ^- for the upper fluid and lower fluid, respectively. If the wave speed U on the interface

satisfies

$$U = \Big((\rho^- - \rho^+)gh/(\rho^-(1+\gamma_1\epsilon))\Big)^{1/2}$$

with $\gamma_1 < 0$ a fixed constant and $\epsilon > 0$ a small parameter, then there is a solitary-wave solution for the exact equations governing the two-fluid flow. Assume $h + \eta^*(x^*)$ is the distance between the rigid bottom and the interface. Then the form of the interface for the solitary-wave solution satisfies

$$h + \eta^*(x^*) = h + \frac{\epsilon a \mu^2 h^3}{(\epsilon x^*)^2 + (h\mu)^2} + \epsilon Y^*(\epsilon x^*/h),$$

where $a = -(4\gamma_1/3), \mu = -(\rho^+/\rho^-\gamma_1)$,

$$\sum_{i=0}^{s} \sup_{x\in\mathbf{R}} |(1+x^2)(d^iY^*(x)/dx^i)| \le K\epsilon,$$

and K is independent of ϵ but may depend on s. Therefore the solitary-wave solution (2) of the Benjamin-Ono equation (1) is the first-order approximation of a solitary-wave solution of the exact equations.

The idea of the proof can be summarized as follows. Since the interface of the two fluids is part of the solution, the domain of the fluids is unknown. We use Levi-Civita's method to transform the unknown domain into a fixed domain using velocity potential and stream function as independent variables. The main advantage of Levi-Civita's transformation is to put all the nonlinear terms in the boundary conditions rather than in the differential equations. It has been shown that such transformation is one to one and onto if the amplitude of the solutions is small [20]. Then we use Fourier transform to invert the differential equations with nonlinear boundary conditions into integral equations. Thus finding a solution of the differential equations becomes looking for a fixed point of corresponding integral operators. However, the kernels in the integral operators have a singularity. In order to remove the singularity, a solvability condition must be satisfied. Next we introduce an unknown function and obtain an integro-differential equation for this function from the solvability condition. By using some techniques in operator theory, we are able to transform the integro-differential equation into an integral equation. By taking advantage of the fact that the nonlinear terms only appear at the boundaries and obtaining the estimates of the integral operators, we can show that the operators are contraction

in certain Banach spaces. The contraction mapping theorem implies the existence of the solution. We note here that the special forms of the equations in the two-fluid flow are important to our approach. The method also works without many changes for a two-fluid system with finite height in upper fluid and infinite depth in lower fluid, which has been studied by Amick [1] (it was known by the author after the major work in this paper was completed). Amick used a different formulation of the problem and a different approach as well to prove the existence of the solitary wave solutions. Kirchgässner [15] also considered surface waves on a layer of fluid with infinite depth. However, we note that the argument given here does not apply to the proof of the existence of solitary waves in continuously stratified fluids with infinite depths, which were included in [4], and further research is needed for such cases.

This paper is organized as follows. In Section 2, all the variables are nondimensionalized and Levi-Civita's method is used to transform the governing equations in an unknown domain into equations in a fixed domain. In Section 3, a formal asymptotic expansion of solutions is used and solitary-wave solutions are formally derived using a different method from the one used in [4]. In Section 4, the equations in Section 2 are transformed into integral equations and a long-wave scaling for the horizontal variable is assumed. A solvability condition is introduced to remove the singularity of the kernels in the integral operators. In Section 5, the estimates of the integral operators are obtained using properties of the Fourier transform in $L^2(\mathbf{R})$. Finally, the existence of solitary-wave solutions is proved in Section 6.

2. Formulation

In this section, we use a formulation devised by Levi-Civita for treating the problem of existence of waves of finite amplitude [14, 20]. The method is motivated by the desire to reformulate the problem in terms of the velocity potential φ^* and stream function ψ^* as independent variables in order to work in a fixed domain in stead of in the partially unknown domain in the physical plane. We consider an irrotational flow of two immiscible, inviscid, and incompressible fluids of different but constant densities bounded by a horizontal rigid bottom and without upper boundary. There exists an interface between

these two fluids and there is a wave at the interface moving with a constant speed U. In the reference to a coordinate system moving with the wave at the same speed, the flow can be regarded as a steady flow. Let $y^* = 0$ be the interface when x^* goes to infinity, $y^* = -h$ be the rigid bottom with $h > 0$, and $y^* = \eta^*(x^*)$ be the interface. Then the upper fluid with density ρ^+ is in $\eta^*(x^*) < y^* < +\infty$ and the lower fluid with density ρ^- is in $-h < y^* < \eta^*(x^*)$, respectively.

Since the flow is irrotational and the fluid is incompressible, we can introduce the velocity potential φ^*, the stream function ψ^*, and the complex velocity potential $\chi^*(x^*, y^*) \equiv \chi^*(z^*)$ with $\chi^* = \varphi^* + i\psi^*$, $z^* = x^* + iy^*$ in $-h < y^* < \eta^*(x^*)$ and $\eta^*(x^*) < y^* < +\infty$. The complex velocity $w^* = d\chi^*/dz^*$ is given by $w^* = u^* - iv^*$, where u^*, v^* are the velocity components. The corresponding dimensionless quantities will be

$$z = z^*/h, \quad w = w^*/U, \quad \rho = \rho^+/\rho^- < 1, \quad \chi = \varphi_1 + i\psi_1 = \chi^*/(hU) \tag{3}$$

so that $d\chi/dz = w$. Furthermore we introduce a nondimensional parameter

$$\gamma = (1-\rho)gh/U^2\,, \tag{4}$$

where $(1/\gamma)^{1/2}$ is so-called nondimensional wave speed. Therefore ψ_1 is a constant along the interface and for the sake of convenience choose $\psi_1 = 0$ at the interface. The bottom corresponds to $\psi_1 = -1$ since the total flow over a curve extending from the bottom to the interface is Uh. In the following we use f^+, f^- to denote a quantity f in the upper- and lower-layer fluid, respectively. The boundary conditions are formulated as follows:

$$v = -\operatorname{Im} w = 0 \quad \text{at} \ \ \psi_1 = -1, \tag{5}$$

$$\gamma y + (1/2)|w^-|^2 - (\rho/2)|w^+|^2 = constant, \tag{6}$$

$$y^+ = y^- \qquad \text{at} \ \ \psi_1 = 0, \tag{7}$$

where the second condition is from the Bernoulli's equation on the interface, w^+, y^+ and w^-, y^- at $\psi_1 = 0$ are denoted the values of w, y as $\psi_1 \to 0^{\pm}$. At the infinity, we have

$$w^{\pm} \to 1 \qquad \text{when} \ \ |x| \to +\infty\,. \tag{8}$$

Therefore the complex potential function $\chi(z)$ maps the physical plane (i.e. the z-plane) onto the strip regions $-1 < \psi_1 < 0, 0 < \psi_1 < +\infty$ in the χ-plane. The mapping is one to one and the inverse mapping function $z(\chi)$ exists as we can easily see later. Taking χ as independent variable, we then determine the analytic function $w(\chi)$ in $-1 < \psi_1 < 0, 0 < \psi_1 < +\infty$ from the boundary conditions.

It is convenient to simplify the form of the boundary conditions by following Levi-Civita's idea [14, 20] to replace the dependent variable w by $\hat{\theta} + i\hat{\lambda}$ through an equation

$$w = \exp\Big(-i(\hat{\theta} + i\hat{\lambda})\Big). \tag{9}$$

Therefore $|w| = e^{\hat{\lambda}}$ and $\hat{\theta} = \arg \overline{w}$, where $|w|$ is the magnitude of the velocity vector and $\hat{\theta}$ is the inclination relative to the x-axis. Now we formulate the conditions for the determination of $\hat{\theta}$ and $\hat{\lambda}$ in the φ_1-ψ_1 plane. Condition (5) is reduced to

$$\hat{\theta} = 0 \qquad \text{at} \ \ \psi_1 = -1\,. \tag{10}$$

By differentiating (6) and (7) with respect to φ_1 along $\psi_1 = 0$, we have

$$\begin{aligned}
&\gamma \frac{\partial y^-}{\partial \varphi_1} + |w^-| \frac{\partial |w^-|}{\partial \varphi_1} - \rho |w^+| \frac{\partial |w^+|}{\partial \varphi_1} = 0, \\
&\frac{\partial y^-}{\partial \varphi_1} = \frac{\partial y^+}{\partial \varphi_1} \qquad \text{at} \ \ \psi_1 = 0\,.
\end{aligned}$$

From $dz/d\chi = (u + iv)|w|^{-2}$ and the analyticity of $\theta + i\lambda$, we obtain $(\partial y/\partial \varphi_1) = v|w|^{-2}$ and $(\partial \hat{\lambda}/\partial \varphi_1) = -(\partial \hat{\theta}/\partial \psi_1)$. Thus at $\psi_1 = 0$ (6) and (7) become

$$e^{2\hat{\lambda}^-} \frac{\partial \hat{\theta}^-}{\partial \psi_1} - \rho e^{2\hat{\lambda}^+} \frac{\partial \hat{\theta}^+}{\partial \varphi_1} - \gamma e^{-\hat{\lambda}^-} \sin \hat{\theta}^- = 0, \tag{11}$$

$$e^{-\hat{\lambda}^-} \sin \hat{\theta}^- = e^{-\hat{\lambda}^+} \sin \hat{\theta}^+\,. \tag{12}$$

When $|\varphi_1| \to +\infty$,

$$\hat{\theta} + i\hat{\lambda} \to 0\,. \tag{13}$$

Since $\hat{\theta} + i\hat{\lambda}$ is an analytic function of $\varphi_1 + i\psi_1$, $\hat{\theta}$ and $\hat{\lambda}$ satisfy the Cauchy-Riemann equations

$$\hat{\theta}_{\varphi_1} = \hat{\lambda}_{\psi_1}, \quad \hat{\lambda}_{\varphi_1} = -\hat{\theta}_{\psi_1} \quad \text{in} \ \ -1 < \psi_1 < 0, \quad 0 < \psi_1 < +\infty. \tag{14}$$

In this paper we are interested in finding bounded nontrivial solutions $(\hat{\theta}, \hat{\lambda})$ of (14) with the boundary conditions (10) to (13) where $\hat{\lambda}$ is even and $\hat{\theta}$ is odd in φ_1.

3. Formal derivation of solitary wave solutions

Although a formal solution has been obtained by Benjamin [4] and many others in different ways, we give another derivation here, which not only gives the first order approximation but also leads to higher order approximations.

To obtain an approximate solution of (10) to (14), we use a new parameter $\varphi = \epsilon\varphi_1$ to stretch the horizontal variable φ_1. Then we also strech the vertical variable by $\psi = \epsilon\psi_1$ in the upper fluid, since the vertical length is infinite in the upper fluid and no long-wave approximation can be assumed, and let the vertical variable in the lower fluid be the same, i.e. $\psi = \psi_1$. Assume that $\hat{\theta}, \hat{\lambda}$ and γ have the asymptotic expansion of the following forms,

$$
\left.\begin{aligned}
\hat{\theta}^+ &= \epsilon^2\theta_1^+(\varphi,\psi) + \epsilon^3\theta_2^+(\varphi,\psi) + \cdots, \\
\hat{\lambda}^+ &= \epsilon^2\lambda_1^+(\varphi,\psi) + \epsilon^3\lambda_2^+(\varphi,\psi) + \cdots,
\end{aligned}\right\} \quad \text{in } 0 < \psi < +\infty,
$$

$$
\left.\begin{aligned}
\hat{\theta}^- &= \epsilon^2\theta_1^-(\varphi,\psi) + \epsilon^3\theta_2^-(\varphi,\psi) + \cdots, \\
\hat{\lambda}^- &= \epsilon\lambda_1^-(\varphi,\psi) + \epsilon^2\lambda_2^-(\varphi,\psi) + \cdots,
\end{aligned}\right\} \quad \text{in } -1 < \psi < 0,
$$

$$
\gamma = \gamma_0 + \epsilon\gamma_1 \,. \tag{15}
$$

In terms of φ, ψ the equation (14) becomes

$$
\begin{aligned}
&\hat{\lambda}_\psi^+ = \hat{\theta}_\varphi^+, \quad \hat{\theta}_\psi^+ = -\hat{\lambda}_\varphi^+ \quad \text{in } 0 < \psi < +\infty, \\
&\hat{\lambda}_\psi^- = \epsilon\hat{\theta}_\varphi^-, \quad \hat{\theta}_\psi^- = -\epsilon\hat{\lambda}_\varphi^- \quad \text{in } -1 < \psi < 0 \,.
\end{aligned}
$$

Substitution of the asymptotic forms for $\hat{\theta}$ and $\hat{\lambda}$ yields different order of the approximations for the equations and boundary conditions. The first-order approximation is

$$
\begin{aligned}
&\lambda_{1\psi}^+ = \theta_{1\varphi}^+, \quad \theta_{1\psi}^+ = -\lambda_{1\varphi}^+ \quad \text{in } 0 < \psi < +\infty, \\
&\lambda_{1\psi}^- = 0, \quad \theta_{1\psi}^- = -\lambda_{1\varphi}^- \quad \text{in } -1 < \psi < 0\,, \\
&\frac{\partial\theta_1^-}{\partial\psi} - \gamma_0\theta_1^- = 0, \quad \theta_1^- = \theta_1^+ \quad \text{at } \psi = 0, \\
&\theta_1^- = 0 \qquad \text{at } \psi = -1\,.
\end{aligned}
$$

In order to have bounded nontrivial solutions $\theta_1^\pm, \lambda_1^\pm$, we need

$$\gamma_0 = 1, \quad \theta_1^-(\varphi,\psi) = S_\varphi(\varphi)(\psi+1), \quad \lambda_1^- = -S(\varphi),$$
$$\theta_1^+ = \overset{\vee}{F}\left[e^{-|k|\psi}\,\hat{F}\,[S_\varphi(\varphi)]\right], \quad \lambda_1^+ = \overset{\vee}{F}\left[(ik)^{-1}|k|e^{-|k|\psi}\,\hat{F}\,[S_\varphi(\varphi)]\right],$$

where $S(\varphi)$ is even and to be determined, and

$$\hat{F}\,[f](k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx} f(x)dx,$$
$$\overset{\vee}{F}\,[f](\varphi) = \int_{-\infty}^{+\infty} e^{-ik\varphi} f(k)dk \tag{16}$$

are the Fourier and inverse-Fourier transforms. The properties of these transforms can be founded in [21, 22].

To find an equation for $S(\varphi)$, we need second order approximation of the equations,

$$\lambda_{2\psi}^+ = \theta_{2\varphi}^+, \quad \theta_{2\psi}^+ = -\lambda_{2\varphi}^+ \quad \text{in } \ 0 < \psi < +\infty,$$
$$\lambda_{2\psi}^- = 0, \quad \theta_{2\psi}^- = -\lambda_{2\varphi}^- \quad \text{in } \ -1 < \psi < 0\,,$$
$$\frac{\partial\theta_2^-}{\partial\psi} - \gamma_0\theta_2^- = \gamma_1\theta_1^- - 2\lambda_1^-\frac{\partial\theta_1^-}{\partial\psi} + \rho\frac{\partial\theta_1^+}{\partial\psi} - \gamma_0\lambda_1^-\theta_1^-,$$
$$\theta_2^- - \lambda_1^-\theta_1^- = \theta_2^+ \quad \text{at } \ \psi = 0,$$
$$\theta_2^- = 0 \qquad \text{at } \ \psi = -1\,.$$

This is a nonhomogeneous boundary value problem. In order to have a solution $(\lambda_2^-, \theta_2^-)$, the nonhomogeneous terms at $\psi = 0$ must satisfy

$$\left(\gamma_1\theta_1^- - 2\lambda_1^-\frac{\partial\theta_1^-}{\partial\psi} + \rho\frac{\partial\theta^+}{\partial\psi} - \gamma_0\lambda_1^-\theta_1^-\right)\Bigg|_{\psi=0} = 0\,.$$

By the first order approximation $\theta_1^\pm, \lambda_1^\pm$ and $\gamma_0 = 1$, $S(\varphi)$ must satisfy

$$\gamma_1 S_\varphi(\varphi) + 3S(\varphi)S_\varphi(\varphi) + \rho\,\overset{\vee}{F}\left[-|k|\,\hat{F}\,[S_\varphi(\varphi)]\right](\varphi) = 0.$$

Since we are looking for solutions decay to zero at infinity, $S(\varphi)$ satisfies

$$\rho\,\overset{\vee}{F}\left[|k|\,\hat{F}\,[S(\varphi)]\right](\varphi) - \gamma_1 S(\varphi) - (3/2)S^2(\varphi) = 0, \tag{17}$$

which is so-called Benjamin-Ono equation [4, 19]. This equation has a solitary wave solution,

$$S(\varphi) = \frac{a\mu^2}{\varphi^2 + \mu^2}, \tag{18}$$

where $a = -(4\gamma_1/3), \mu = -(\rho/\gamma_1)$ with $\gamma_1 < 0$. The derivation of solution (18) from (17) can be found in [4]. Therefore the first-order approximation in (15) can be obtained. The first-order approximation of the wave profile at the interface is

$$y = \epsilon S(\varphi) + O(\epsilon^2) = \frac{a\mu^2\epsilon}{\varphi^2 + \mu^2} + O(\epsilon^2). \tag{19}$$

Since $a > 0$ with $\gamma_1 < 0$, this is an elevation solitary wave. This formal procedure can be easily carried over to find the higher order approximations.

In the following sections, we shall show rigorously that there exists a solitary wave solution of (10) to (14) and (19) is the first-order approximation of this solution at the interface $\psi_1 = 0$.

4. Transformations and Banach spaces

In this section, we transform (10) to (14) into integral equations. Let $\gamma = 1 + \gamma_1\epsilon$ with $\gamma_1 < 0$ and $\epsilon > 0$.

First we consider the equations in the domain $0 < \psi_1 < +\infty$. The equation (14) implies

$$\hat{\theta}^+_{\varphi_1} = \hat{\lambda}^+_{\psi_1}, \quad \hat{\lambda}^+_{\varphi_1} = -\hat{\theta}^+_{\psi_1}, \qquad \hat{\theta}^+_{\psi_1\psi_1} + \hat{\theta}^+_{\varphi_1\varphi_1} = 0. \tag{20}$$

If $\hat{\lambda}^+, \hat{\lambda}^-$ and $\hat{\theta}^-$ are small enough, (12) can be written as

$$\hat{\theta}^+ = \arcsin(e^{\hat{\lambda}^+ - \hat{\lambda}^-} \sin\hat{\theta}^-) = b(\varphi_1) \quad \text{at} \;\; \psi_1 = 0. \tag{21}$$

By applying the Fourier transform on (20), we have

$$(\hat{F}[\hat{\theta}^+])_{\psi_1\psi_1} - |k|^2 \hat{F}[\hat{\theta}^+] = 0,$$

where $\hat{F}[f]$ is defined in (16). If $\hat{F}[\hat{\theta}^+]$ is finite as $\psi_1 \to +\infty$,

$$\hat{F}[\hat{\theta}^+](k, \psi_1) = e^{-|k|\psi_1} \hat{F}[b(\varphi_1)],$$

which gives

$$\hat{\theta}^+(\varphi_1,\psi_1) = \overset{\vee}{F}\left[e^{-|k|\psi_1}\ \hat{F}\left[\arcsin(e^{\hat{\lambda}^+-\hat{\lambda}^-}\sin\hat{\theta}^-)\Big|_{\psi_1=0}\right]\right]. \tag{22}$$

By (20) again, we have

$$\hat{\lambda}^+(\varphi_1,\psi_1) = \overset{\vee}{F}\left[(i|k|/k)e^{-|k|\psi_1}\ \hat{F}\left[\arcsin(e^{\hat{\lambda}^+-\hat{\lambda}^-}\sin\hat{\theta}^-)\Big|_{\psi_1=0}\right]\right]. \tag{23}$$

Note that (22) and (23) hold in $0<\psi_1<+\infty$.

Next we study the equations (10) to (14) in the domain $-1<\psi_1<0$. The equations are

$$\hat{\lambda}^-_{\psi_1} = \hat{\theta}^-_{\varphi_1},\quad \hat{\theta}^-_{\psi_1} = -\hat{\lambda}^-_{\varphi_1},\quad \hat{\theta}^-_{\psi_1\psi_1}+\hat{\theta}^-_{\varphi_1\varphi_1}=0 \quad \text{in} \quad -1<\psi_1<0, \tag{24}$$

$$\begin{aligned}\frac{\partial\hat{\theta}^-}{\partial\psi_1}-\hat{\theta}^- &= \gamma_1\epsilon\hat{\theta}^- + \rho e^{2(\hat{\lambda}^+-\hat{\lambda}^-)}\frac{\partial\hat{\theta}^+}{\partial\psi_1}\\ &\quad + (1+\gamma_1\epsilon)(e^{-3\hat{\lambda}^-}\sin(\hat{\theta}^-)-\hat{\theta}^-) \qquad \text{at} \;\; \psi_1=0,\end{aligned} \tag{25}$$

$$\hat{\theta}^- = 0 \qquad\qquad \text{at} \quad \psi_1=-1. \tag{26}$$

From (22), we can rewrite (25) as

$$\begin{aligned}\frac{\partial\hat{\theta}^-}{\partial\psi_1}-\hat{\theta}^-+\rho\ \overset{\vee}{F}\left[|k|\ \hat{F}\left[\hat{\theta}^-\right]\right] &= \gamma_1\epsilon\hat{\theta}^-\\ &\quad + (1+\gamma_1\epsilon)(e^{-3\hat{\lambda}^-}\sin(\hat{\theta}^-)-\hat{\theta}^-) + \rho(1-e^{2(\hat{\lambda}^+-\hat{\lambda}^-)})\ \overset{\vee}{F}\left[|k|\ \hat{F}\left[\hat{\theta}^-\right]\right]\\ &\quad + \rho e^{2(\hat{\lambda}^+-\hat{\lambda}^-)}\ \overset{\vee}{F}\left[|k|\ \hat{F}\left[\hat{\theta}^- - \arcsin(e^{\hat{\lambda}^+-\hat{\lambda}^-}\sin\hat{\theta}^-)\right]\right]\\ &= f_1(\hat{\theta}^-,\hat{\lambda}^-,\hat{\lambda}^+).\end{aligned} \tag{27}$$

By using the Fourier transform, (24), (26) and (27), we obtain

$$\hat{\theta}^-(\varphi_1,\psi_1) = \overset{\vee}{F}\left[\frac{\sinh(|k|(\psi_1+1))\ \hat{F}[f_1]}{|k|\cosh|k|-\sinh|k|+\rho|k|\sinh|k|}\right], \tag{28}$$

$$\hat{\lambda}^-(\varphi_1,\psi_1) = \overset{\vee}{F}\left[\frac{|k|\cosh(|k|(\psi_1+1))\ \hat{F}[f_1]}{ik(|k|\cosh|k|-\sinh|k|+\rho|k|\sinh|k|)}\right]. \tag{29}$$

Thus the equations (10) to (14) are transformed into integral equations (22), (23), (28) and (29), where f_1 is defined in (27).

A crucial step in our procedure is that we make the same change of independent variables

$$\varphi = \epsilon\varphi_1, \qquad \psi = \psi_1 \,, \tag{30}$$

which we made in Section 3. In accordance with the development of Section 3, we put

$$\theta^{\pm} = \epsilon^{-2}\hat{\theta}^{\pm}, \qquad \lambda^{\pm} = \epsilon^{-1}\hat{\lambda}^{\pm} . \tag{31}$$

Then by changing the integration variables in (22), (23), (28) and (29), these equations become: In $0 < \psi < +\infty$,

$$\theta^{+}(\varphi,\psi) = \overset{\vee}{F}\left[e^{-\epsilon|k|\psi}\,\hat{F}\left[\epsilon^{-2}\arcsin\left(e^{\epsilon(\lambda^{+}-\lambda^{-})}\sin(\epsilon^{2}\theta^{-})\Big|_{\psi=0}\right)\right]\right], \tag{32}$$

$$\lambda^{+}(\varphi,\psi) = \overset{\vee}{F}\left[(i|k|/k)e^{-\epsilon|k|\psi}\,\hat{F}\left[\epsilon^{-1}\arcsin\left(e^{\epsilon(\lambda^{+}-\lambda^{-})}\sin(\epsilon^{2}\theta^{-})\Big|_{\psi=0}\right)\right]\right]; \tag{33}$$

In $-1 < \psi < 0$,

$$\theta^{-}(\varphi,\psi) = \overset{\vee}{F}\left[\frac{\sinh(\epsilon|k|(\psi+1))\,\hat{F}[f_2]}{\rho\epsilon|k|\sinh\epsilon|k| + \epsilon|k|\cosh\epsilon|k| - \sinh\epsilon|k|}\right], \tag{34}$$

$$\lambda^{-}(\varphi,\psi) = \overset{\vee}{F}\left[\frac{|k|\cosh(\epsilon|k|(\psi+1))\,\hat{F}[\epsilon f_2]}{ik(\rho\epsilon|k|\sinh\epsilon|k| + \epsilon|k|\cosh\epsilon|k| - \sinh\epsilon|k|)}\right], \tag{35}$$

where

$$\begin{aligned} f_2(\theta^{-},\lambda^{-},\lambda^{+}) &= \epsilon^{-2}f_1(\epsilon^{2}\theta^{-},\epsilon\lambda^{-},\epsilon\lambda^{+}) \\ &= [\gamma_1\epsilon\theta^{-} + (1+\gamma_1\epsilon)(\epsilon^{-2}e^{-3\epsilon\lambda^{-}}\sin(\epsilon^{2}\theta^{-}) - \theta^{-})] \\ &\quad + \rho e^{2(\epsilon\lambda^{+}-\epsilon\lambda^{-})}\,\overset{\vee}{F}\left[\epsilon|k|\,\hat{F}\left[\theta^{-} - \epsilon^{-2}\arcsin(e^{\epsilon\lambda^{+}-\epsilon\lambda^{-}}\sin(\epsilon^{2}\theta^{-}))\right]\right] \\ &\quad + \rho(1-e^{2(\epsilon\lambda^{+}-\epsilon\lambda^{-})})\,\overset{\vee}{F}\left[\epsilon|k|\,\hat{F}\left[\theta^{-}\right]\right] \\ &= I + II + III . \end{aligned}$$

Notice that when ϵ or k is small, the denominators in the inverse-Fourier transforms of (34) and (35) have a zero point of order $\epsilon^2 k^2$. Therefore we have to single out this singularity

in the integrals. Write

$$
\begin{aligned}
\frac{\sinh(\epsilon|k|(\psi+1))}{\rho\epsilon|k|\sinh\epsilon|k|+\epsilon|k|\cosh\epsilon|k|-\sinh\epsilon|k|} &= \frac{\psi+1}{\rho\epsilon|k|} \\
&- \frac{(\psi+1)(\epsilon|k|\cosh(\epsilon|k|)-\sinh(\epsilon|k|))}{\rho\epsilon|k|(\rho\epsilon|k|\sinh\epsilon|k|+\epsilon|k|\cosh\epsilon|k|-\sinh\epsilon|k|)} \\
&+ \frac{\sinh(\epsilon|k|(\psi+1))-(\psi+1)\sinh(\epsilon|k|)}{\rho\epsilon|k|\sinh\epsilon|k|+\epsilon|k|\cosh\epsilon|k|-\sinh\epsilon|k|} \\
&\stackrel{\text{def}}{=} k_1(k)(\psi+1)+k_2(k)(\psi+1)+k_3(k,\psi),
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
\frac{|k|\cosh(\epsilon|k|(\psi+1))}{ik(\rho\epsilon|k|\sinh\epsilon|k|+\epsilon|k|\cosh\epsilon|k|-\sinh\epsilon|k|)} &= \frac{1}{i\rho\epsilon^2k|k|} \\
&- \frac{(\epsilon|k|\cosh(\epsilon|k|)-\sinh(\epsilon|k|))}{i\rho\epsilon^2k|k|(\rho\epsilon|k|\sinh\epsilon|k|+\epsilon|k|\cosh\epsilon|k|-\sinh\epsilon|k|)} \\
&+ \frac{(\epsilon|k|\cosh(\epsilon|k|(\psi+1))-\sinh(\epsilon|k|))}{ik\epsilon(\rho\epsilon|k|\sinh\epsilon|k|+\epsilon|k|\cosh\epsilon|k|-\sinh\epsilon|k|)} \\
&\stackrel{\text{def}}{=} (1/i\epsilon k)\Big(k_1(k)+k_2(k)+k_{3\psi}(k,\psi)\Big).
\end{aligned} \tag{37}
$$

Therefore (34) and (35) become

$$
\begin{aligned}
\theta^-(\varphi,\psi) &= (\psi+1)\Big(\stackrel{\vee}{F}\Big[(1/\rho\epsilon|k|)\stackrel{\wedge}{F}[f_2]\Big]+\stackrel{\vee}{F}\Big[k_2(k)\stackrel{\wedge}{F}[f_2]\Big]\Big) \\
&\quad + \stackrel{\vee}{F}\Big[k_3(k,\psi)\stackrel{\wedge}{F}[f_2]\Big], \qquad (38) \\
\lambda^-(\varphi,\psi) &= \stackrel{\vee}{F}\Big[(1/i\rho\epsilon k|k|)\stackrel{\wedge}{F}[f_2]\Big]+\stackrel{\vee}{F}\Big[(1/ik)k_2(k)\stackrel{\wedge}{F}[f_2]\Big] \\
&\quad + \stackrel{\vee}{F}\Big[(1/ik)k_{3\psi}(k,\psi)\stackrel{\wedge}{F}[f_2]\Big]. \qquad (39)
\end{aligned}
$$

Note that $k_3(k,\psi)$ is finite as $\epsilon k \to 0$. Therefore we need to remove first two terms in the right hand sides of (38) and (39). Let

$$
\lambda^- = -\eta(\varphi)+\zeta \quad \text{and then} \quad \theta^- = (\psi+1)\eta_\varphi(\varphi)+\xi\,. \tag{40}
$$

If we assume

$$
-\eta = \stackrel{\vee}{F}\Big[(1/i\rho\epsilon k|k|)\stackrel{\wedge}{F}[f_2]\Big]+\stackrel{\vee}{F}\Big[(1/ik)k_2(k)\stackrel{\wedge}{F}[f_2]\Big], \tag{41}
$$

which is called solvability condition, then (38) and (39) are changed into

$$\xi = \overset{\vee}{F} \left[k_3(k,\psi) \hat{F} [f_2] \right], \tag{42}$$

$$\zeta = \overset{\vee}{F} \left[(1/ik) k_{3\psi}(k,\psi) \hat{F} [f_2] \right]. \tag{43}$$

By the definition, $f_2(\theta^-, \lambda^-, \lambda^+)$ can be written as

$$\begin{aligned} f_2(\theta^-, \lambda^-, \lambda^+) &= f_2(\eta_\varphi + \xi, -\eta + \zeta, \lambda^+)\Big|_{\psi=0} \\ &= \gamma_1 \epsilon \theta^- - 3\epsilon \lambda^- \theta^- + \epsilon^2 \mathcal{R}(\theta^-, \lambda^-, \lambda^+) \\ &= \gamma_1 \epsilon \eta_\varphi + 3\epsilon \eta \eta_\varphi + \gamma_1 \epsilon \xi - 3\epsilon(\zeta \eta_\varphi - \xi \eta + \zeta \xi) \\ &\qquad + \epsilon^2 \mathcal{R}(\theta^-, \lambda^-, \lambda^+) \\ &= \gamma_1 \epsilon \eta_\varphi + 3\epsilon \eta \eta_\varphi + \epsilon \tilde{f}_2(\eta, \xi, \zeta, \lambda^+). \end{aligned} \tag{44}$$

Thus by taking the Fourier transform on both sides of (41), we can obtain

$$-\rho\epsilon|k|ki \hat{F} [\eta] = (1 + \rho\epsilon|k|k_2(k)) \hat{F} [f_2],$$

which implies

$$\begin{aligned} \overset{\vee}{F} \left[\rho|k|(1+\rho\epsilon|k|k_2(k))^{-1} \hat{F} [\eta] \right]_\varphi &= (1/\epsilon) f_2(\theta^-, \lambda^-, \lambda^+) \\ &= \gamma_1 \eta_\varphi + (3/2)(\eta^2)_\varphi + \tilde{f}_2(\eta, \xi, \zeta, \lambda^+). \end{aligned} \tag{45}$$

If we let $\epsilon = 0$ and $\tilde{f}_2 = 0$, (45) becomes (17) and it has a solution

$$S(\varphi) = \frac{a\mu^2}{\varphi^2 + \mu^2}, \tag{46}$$

where $a = -(4\gamma_1/3), \mu = -(\rho/\gamma_1)$. Assume

$$\eta = S(\varphi) + \omega(\varphi). \tag{47}$$

Then substitution of (47) into (45) yields

$$\begin{aligned} &\rho \left(\overset{\vee}{F} \left[|k|(1+\rho\epsilon|k|k_2(k))^{-1} \hat{F} [\omega] \right] \right)_\varphi - \gamma_1 \omega_\varphi - 3(S\omega)_\varphi \\ &\quad = (3/2)(\omega^2)_\varphi + \rho \left(\overset{\vee}{F} \left[|k|^2 \rho\epsilon k_2(k)(1+\rho\epsilon|k|k_2(k))^{-1} \hat{F} [S(\varphi)] \right] \right)_\varphi + \tilde{f}_2(\eta, \xi, \zeta, \lambda^+). \end{aligned}$$

By applying the integration once, we have

$$\begin{aligned}
&\rho \overset{\vee}{F}\left[|k|(1+\rho\epsilon|k|k_2(k))^{-1}\hat{F}[\omega]\right]-\gamma_1\omega-3S\omega \\
&\quad=(3/2)\omega^2+\rho\overset{\vee}{F}\left[|k|^2\rho\epsilon k_2(k)(1+\rho\epsilon|k|k_2(k))^{-1}\hat{F}[S(\varphi)]\right]+\int_{\varphi}^{+\infty}\tilde{f}_2(\eta,\xi,\zeta,\lambda^+)d\varphi \\
&\quad=g_1(\omega,\xi,\zeta,\lambda^+)\,. \qquad (48)
\end{aligned}$$

In the following, we shall show that (32), (33), (42), (43) and (48) have a solution $(\omega,\xi,\zeta,\theta^+,\lambda^+)$ in some Banach space for small $\epsilon>0$, where ω,ζ,λ^+ are even and ξ,θ^+ are odd in φ.

Before proving the existence, we define some Banach spaces for later use. Let $H^s(\mathbf{R})$ be the usual Sobolev spaces on $\mathbf{R}=(-\infty,+\infty)$ with $s\geq 0$ [22]. Define

$$W^s=\{f(\varphi)\in H^s(\mathbf{R})\;\Big|\;\varphi f(\varphi)\in H^s(\mathbf{R})\;\text{ and }\;\|f\|_{W^s}=\|(1+\varphi^2)^{1/2}f\|_{H^s(\mathbf{R})}\},$$

$$B^s=\{f(\varphi,\psi)\in L^2(\mathbf{R}\times[-1,0])\;\Big|\;\text{for any }\psi\in[-1,0],f(\varphi,\psi)\in W^s,\text{ and}$$
$$\|f\|_{B^s}=\sup_{-1\leq\psi\leq 0}\|f(\varphi,\psi)\|_{W^s}<+\infty\}\,.$$

We use W^s_e,B^s_e,H^s_e or W^s_o,B^s_o,H^s_o to denote the corresponding spaces with even functions or odd functions in φ. Note that $f(\varphi)\in W^s$ if and only if $\hat{F}[f](k)$ is differentiable in k and

$$(1+k^2)^{s/2}\hat{F}[f](k)\in L^2(\mathbf{R}),\qquad (1+k^2)^{s/2}\frac{d}{dk}\hat{F}[f](k)\in L^2(\mathbf{R})\,.$$

Also by using the Sobolev embedding theorem, we have

Lemma 1. If $u,v\in H^s(\mathbf{R})$ for $s\geq 1$, then $uv\in H^s(\mathbf{R})$ and

$$\|uv\|_{H^s(\mathbf{R})}\leq K\|u\|_{H^s(\mathbf{R})}\|v\|_{H^s(\mathbf{R})}\,,$$

where K is a fixed constant.

5. Estimates of the operators

First let us consider (32) and (33). Rewrite these equations into

$$\theta^+(\varphi,\psi)=\overset{\vee}{F}\left[e^{-\epsilon|k|\psi}\hat{F}[S_\varphi]\right]+\overset{\vee}{F}\left[e^{-\epsilon|k|\psi}\hat{F}[\omega_\varphi+\xi(\varphi,0)]\right]$$

$$
\begin{aligned}
&+ \overset{\vee}{F}\left[e^{-\epsilon|k|\psi}\ \hat{F}\left[\left(\epsilon^{-2}\arcsin(e^{\epsilon(\lambda^+-\lambda^-)}\sin\epsilon^2\theta^-)-\theta^-\right)\Big|_{\psi=0}\right]\right] \\
&\overset{\text{def}}{=} L_1(S)+L_2(\omega,\xi)+L_3(\omega,\xi,\zeta,\lambda^+), \qquad (49)\\
\lambda^+(\varphi,\psi) &= \overset{\vee}{F}\left[(i|k|\epsilon/k)e^{-\epsilon|k|\psi}\ \hat{F}[S_\varphi]\right]+\overset{\vee}{F}\left[(i|k|\epsilon/k)e^{-\epsilon|k|\psi}\ \hat{F}[\omega_\varphi+\xi(\varphi,0)]\right]\\
&+ \overset{\vee}{F}\left[(\epsilon i|k|/k)e^{-\epsilon|k|\psi}\ \hat{F}\left[\left(\epsilon^{-2}\arcsin(e^{\epsilon(\lambda^+-\lambda^-)}\sin\epsilon^2\theta^-)-\theta^-\right)\Big|_{\psi=0}\right]\right]\\
&\overset{\text{def}}{=} M_1(S)+M_2(\omega,\xi)+M_3(\omega,\xi,\zeta,\lambda^+), \qquad (50)
\end{aligned}
$$

where L_3, M_3 are of order ϵ for small $\epsilon > 0$. By using the properties of Fourier transforms, Lemma 1, and the Plancherel equality, we can have that if $\|\xi(\varphi,0)\|_{W_o^s} + \|\zeta(\varphi,0)\|_{W_e^s} + \|\omega_\varphi\|_{W_o^s} + \|\lambda^+(\varphi,0)\|_{W_e^s} \le K$ for $s \ge 2$, then for $0 \le \psi < +\infty$,

$$
\|L_2(\omega,\xi)\|_{H_o^s}+\|M_2(\omega,\xi)\|_{H_e^s}\le K(\|\omega_\varphi\|_{W_o^s}+\|\xi(\varphi,0)\|_{W_o^s})
$$

and

$$
\|L_3(\omega,\xi,\zeta,\lambda^+)\|_{H_o^s}+\|M_3(\omega,\xi,\zeta,\lambda^+)\|_{H_e^s}\le K\epsilon\,.
$$

Since $|k|$ is piecewise differentiable and ω_φ, ξ are odd,

$$
\begin{aligned}
\varphi M_2(\omega,\xi) &= 2\varphi\int_0^{+\infty}\cos(k\varphi)(i\epsilon)e^{-\epsilon|k|\psi}\ \hat{F}[\omega_\varphi+\xi(\varphi,0)]\,dk\\
&= -2\int_0^{+\infty}\sin(k\varphi)(i\epsilon)\left(\frac{d}{dk}e^{-\epsilon|k|\psi}\ \hat{F}[\omega_\varphi+\xi(\varphi,0)]\right)dk\\
&= \int_{-\infty}^{+\infty}e^{-ik\varphi}(\mathrm{sgn}(k))\epsilon\left(\frac{d}{dk}e^{-\epsilon|k|\psi}\ \hat{F}[\omega_\varphi+\xi(\varphi,0)]\right)dk\,,
\end{aligned}
$$

where $\mathrm{sgn}(k) = 1$ for $k > 0$ and $\mathrm{sgn}(k) = -1$ for $k < 0$. Thus for ψ bounded,

$$
\begin{aligned}
&\|\varphi L_2(\omega,\xi)\|_{H_e^s}+\|\varphi M_2(\omega,\xi)\|_{H_o^s}\\
&\quad\le K\Bigg(\left\|(1+k^2)^{s/2}\left(d\left(e^{-\epsilon|k|\psi}\ \hat{F}[\omega_\varphi+\xi(\varphi,0)]\right)/dk\right)\right\|_{L^2(\mathbf{R})}\\
&\qquad+\left\|(1+k^2)^{s/2}\mathrm{sgn}(k)\left(d\left((e^{-\epsilon|k|\psi}\ \hat{F}[\omega_\varphi+\xi(\varphi,0)]\right)/dk\right)\right\|_{L^2(\mathbf{R})}\Bigg)\\
&\quad\le K(\|\omega_\varphi\|_{W_o^s}+\|\xi(\varphi,0)\|_{W_o^s})\,.
\end{aligned}
$$

By using the same argument, we have

$$
\|\varphi L_3(\omega,\xi,\zeta,\lambda^+)\|_{H_e^s}+\|\varphi M_3(\omega,\xi,\zeta,\lambda^+)\|_{H_o^s}\le K\epsilon\,.
$$

Therefore we obtain

Theorem 1. If $\|\omega_\varphi\|_{W^s_o} + \|\xi(\varphi,0)\|_{W^s_o} + \|\zeta(\varphi,0)\|_{W^s_e} + \|\lambda^+(\varphi,0)\|_{W^s_e} \leq K$ with $s \geq 2$, then for $0 \leq \psi \leq K_1 < +\infty$

$$\|L_2(\omega,\xi)\|_{W^s_o} + \|M_2(\omega,\xi)\|_{W^s_e} \leq K(\|\omega_\varphi\|_{W^s_o} + \|\xi(\varphi,0)\|_{W^s_o}),$$
$$\|L_3(\omega,\xi,\zeta,\lambda^+)\|_{W^s_o} + \|M_3(\omega,\xi,\zeta,\lambda^+)\|_{W^s_e} \leq K\epsilon.$$

Next we consider (42) and (43). From the forms of $k_3(k,\psi)$ and $k_{3\psi}(k,\psi)$, it is easy to see that for any small $\epsilon > 0, k \in (-\infty,+\infty)$ and $-1 \leq \psi \leq 0$,

$$|k_3(k,\psi)| + |k_{3\psi}(k,\psi)| + |\epsilon k k_3(k,\psi)| + |(\epsilon k)^{-1} k_{3\psi}(k,\psi)| \leq K_2 < +\infty,$$

where K_2 is a fixed constant. From the definition of f_2 in (34) and (35), we also have

$$\begin{aligned}\overset{\vee}{F}\left[k_3(k,\psi)\ \hat{F}[f_2]\right] =& \overset{\vee}{F}\left[k_3(k,\psi)\ \hat{F}[I]\right] \\ &+ \overset{\vee}{F}\left[k_3(k,\psi)\ \hat{F}[II]\right] + \overset{\vee}{F}\left[k_3(k,\psi)\ \hat{F}[III]\right].\end{aligned}$$

Assume that at $\psi = 0$, $\|\lambda^+\|_{W^s_e} + \|\lambda^-\|_{W^s_e} + \|\theta^-\|_{W^s_o} + \|\theta^+\|_{W^s_o} \leq K$ for $s \geq 2$. By Lemma 1, the forms of I, II and III, and the Plancherel equality, we have that for $-1 < \psi < 0$

$$\left\| \overset{\vee}{F}\left[k_3(k,\psi)\ \hat{F}[I]\right] \right\|_{H^s_o} \leq K\|I\|_{H^s_o} \leq K\epsilon.$$

Also

$$\begin{aligned}\left\| \overset{\vee}{F}\left[k_3(k,\psi)\ \hat{F}[II]\right] \right\|^2_{L^2(\mathbf{R})} &\leq \|k_3(k,\psi)\ \hat{F}[II]\|^2_{L^2(\mathbf{R})} \leq K\|II\|^2_{L^2(\mathbf{R})} \\ &\leq K\left\| \overset{\vee}{F}\left[|k|\epsilon\ \hat{F}\left[\theta^- - \epsilon^{-2}\arcsin(e^{\epsilon(\lambda^+-\lambda^-)}\sin\epsilon^2\theta^-)\right]\right] \right\|^2_{L^2(\mathbf{R})} \\ &\leq K\epsilon^2(\|\theta^-\|^2_{H^1_o} + \|\lambda^-\|^2_{H^1_e} + \|\lambda^+\|^2_{H^1_e}),\end{aligned}$$

and for $s \geq 1$

$$\begin{aligned}\left\|\frac{d^s}{d\varphi^s}\ \overset{\vee}{F}\left[k_3(k,\psi)\ \hat{F}[II]\right] \right\|^2_{L^2(\mathbf{R})} &\leq \|k^s k_3(k,\psi)\ \hat{F}[II]\|^2_{L^2(\mathbf{R})} \\ &\leq K\|k^{s-1}\epsilon^{-1}\ \hat{F}[II]\|^2_{L^2(\mathbf{R})} \leq K\|(1/\epsilon)(d^{s-1}II/d\varphi)\|^2_{L^2(\mathbf{R})} \\ &\leq K\left\|\frac{d^{s-1}}{ds^{s-1}}\ \overset{\vee}{F}\left[|k|\ \hat{F}\left[\theta^- - \epsilon^{-2}\arcsin(e^{\epsilon(\lambda^+-\lambda^-)}\sin\epsilon^2\theta^-)\right]\right] \right\|^2_{L^2(\mathbf{R})} \\ &\leq K\epsilon^2(\|\theta^-\|^2_{H^s_o} + \|\lambda^-\|^2_{H^s_e} + \|\lambda^+\|^2_{H^s_e}).\end{aligned}$$

Thus $\| \overset{\vee}{F} [k_3(k,\psi) \hat{F} [II]] \|_{H_o^s} \leq K\epsilon$. By using similar estimates for III, we can have $\| \overset{\vee}{F} [k_3(k,\psi) \hat{F} [III]] \|_{H_o^s} \leq K\epsilon$. Therefore for $-1 < \psi < 0$

$$\left\| \overset{\vee}{F} \left[k_3(k,\psi) \hat{F} [f_2] \right] \right\|_{H_o^s} \leq K\epsilon.$$

Also note that $|k|$ is piecewise differentiable and the derivatives of k_3, $k_{3\psi}$, $\epsilon k k_3$ and $(1/\epsilon k) k_{3\psi}$ with respect to k are bounded. Therefore using similar arguments, we can have

$$\begin{aligned} \left\| \varphi \overset{\vee}{F} \left[k_3(k,\psi) \hat{F} [f_2] \right] \right\|_{H^s} &\leq K \left\| \overset{\vee}{F} \left[(dk_3(k,\psi)/dk) \hat{F} [f_2] \right] \right\|_{H^s} \\ &+ \left\| \overset{\vee}{F} \left[k_3(k,\psi) \hat{F} [\varphi f_2] \right] \right\|_{H^s} \leq K\epsilon. \end{aligned}$$

Thus $\overset{\vee}{F} [k_3(k,\psi) \hat{F} [f_2]] \in W_o^s$ and

$$\left\| \overset{\vee}{F} \left[k_3(k,\psi) \hat{F} [f_2] \right] \right\|_{W_o^s} \leq K\epsilon.$$

By a similar reason,

$$\begin{aligned} \left\| \overset{\vee}{F} \left[(1/ik) k_{3\psi}(k,\psi) \hat{F} [f_2] \right] \right\|_{L^2(\mathbf{R})} &\leq \| (1/ik\epsilon) k_{3\psi}(k,\psi) \epsilon \hat{F} [f_2] \|_{L^2(\mathbf{R})} \\ &\leq K \| f_2 \|_{L^2(\mathbf{R})} \leq K\epsilon, \end{aligned}$$

and for $s \geq 1$,

$$\begin{aligned} \left\| \frac{d^s}{d\varphi^s} \overset{\vee}{F} \left[(1/ik) k_{3\psi}(k,\psi) \hat{F} [f_2] \right] \right\|_{L^2(\mathbf{R})} &\leq \| k^{s-1} k_{3\psi}(k,\psi) \hat{F} [f_2] \|_{L^2(\mathbf{R})} \\ &\leq K \| (d^{s-1} f_2 / d\varphi^{s-1}) \|_{L^2(\mathbf{R})} \leq K\epsilon. \end{aligned}$$

Thus $\| \overset{\vee}{F} [(1/ik) k_{3\psi}(k,\psi) \hat{F} [f_2]] \|_{H^s} \leq K\epsilon$. By applying the similar argument, we can have $\| \varphi \overset{\vee}{F} [(1/ik) k_{3\psi}(k,\psi) \hat{F} [f_2]] \|_{H^s} \leq K\epsilon$ since $|k|$ is piecewise differentiable. Therefore $\overset{\vee}{F} [(1/ik) k_{3\psi}(k,\psi) \hat{F} [f_2]] \in W_e^s$ and

$$\left\| \overset{\vee}{F} \left[(1/ik) k_{3\psi}(k,\psi) \hat{F} [f_2] \right] \right\|_{W_e^s} \leq K\epsilon.$$

We summarize the above results as follows.

Theorem 2. If $\theta^- = (S+\omega)_\varphi(\psi+1)+\xi \in B_o^s, \lambda^- = -(S+\omega)+\zeta \in B_e^s, \lambda^+(\varphi,0) \in W_e^s$ and $\|\omega\|_{W_e^{s+1}} + \|\xi\|_{B_o^s} + \|\zeta\|_{B_e^s} + \|\lambda^+(\varphi,0)\|_{W_e^s} \le K$ with $s \ge 2$, then

$$\left\| \overset{\vee}{F} \left[k_3(k,\psi) \hat{F}[f_2] \right] \right\|_{B_o^s} \le K\epsilon,$$
$$\left\| \overset{\vee}{F} \left[(1/ik) k_{3\psi}(k,\psi) \hat{F}[f_2] \right] \right\|_{B_e^s} \le K\epsilon,$$

where K is a constant independent of ϵ and ψ.

Finally we study the equation (48). Let

$$N(\omega) = \rho \overset{\vee}{F} \left[|k|(1+\rho\epsilon|k|k_2(k))^{-1} \hat{F}[\omega] \right] - \gamma_1\omega - 3S\omega. \tag{51}$$

Then (48) becomes

$$N(\omega) = g_1(\omega,\xi,\zeta,\lambda^+). \tag{52}$$

We first study the general nonhomogeneous equation

$$N(\omega) = g(\varphi), \qquad \omega_\varphi(0) = 0, \tag{53}$$

where $\omega(\varphi)$ and $g(\varphi)$ are even functions in φ. By applying the Fourier transform on both sides of (53) and using $\hat{F}[S(\varphi)] = (a\mu/2)\exp(-\mu|k|)$, we have

$$\begin{aligned} &\rho|k|(1+\rho\epsilon|k|k_2(k))^{-1} \hat{F}[\omega](k) - \gamma_1 \hat{F}[\omega](k) \\ &\qquad - (3/2)\int_{-\infty}^{+\infty} a\mu e^{-\mu|k-k'|} \hat{F}[\omega](k')dk' = \hat{F}[g](k), \end{aligned} \tag{54}$$

where a and μ are defined in (46). Thus $\hat{F}[\omega](k)$ must satisfy

$$\begin{aligned} &\hat{F}[\omega](k) - \frac{3a\mu}{2(\rho|k|(1+\rho\epsilon|k|k_2(k))^{-1}-\gamma_1)} \int_{-\infty}^{+\infty} e^{-\mu|k-k'|} \hat{F}[\omega](k')dk' \\ &\qquad = \frac{\hat{F}[g](k)}{\rho|k|(1+\rho\epsilon|k|k_2(k))^{-1}-\gamma_1} \overset{\text{def}}{=} g_0(k). \end{aligned} \tag{55}$$

Note that by the definition of $k_2(k)$,

$$\begin{aligned} (1+\rho\epsilon|k|k_2(k))^{-1} &= \left(1 - \frac{\epsilon|k|\cosh\epsilon|k| - \sinh\epsilon|k|}{\rho\epsilon|k|\sinh\epsilon|k| + \epsilon|k|\cosh\epsilon|k| - \sinh\epsilon|k|}\right)^{-1} \\ &= 1 + \frac{\epsilon|k|\cosh\epsilon|k| - \sinh\epsilon|k|}{\rho|k|\epsilon\sinh\epsilon|k|} \ge 1. \end{aligned} \tag{56}$$

Denote the integral operator in (55) by

$$T_\epsilon(u(k)) \stackrel{\text{def}}{=} \frac{3a\mu}{2(\rho|k|(1+\rho\epsilon|k|k_2(k))^{-1}-\gamma_1)}\int_{-\infty}^{+\infty} e^{-\mu|k-k'|}u(k')dk' \tag{57}$$

and $T(u) \stackrel{\text{def}}{=} T_0(u)$. Then (55) becomes

$$\stackrel{\wedge}{F}[\omega] - T_\epsilon(\stackrel{\wedge}{F}[\omega]) = g_0(k)\,. \tag{58}$$

Now consider an integral equation

$$u - T(u) = f, \tag{59}$$

where $f \in L^2(\mathbf{R})$ and u, f are even in φ. Let

$$W(k) \stackrel{\text{def}}{=} -(1/2\mu)\int_{-\infty}^{+\infty} e^{-\mu|k-k'|}u(k')dk'.$$

By (59), $W(k)$ satisfies

$$W_{kk} - \mu^2 W + \frac{3a\mu^2}{\rho|k|-\gamma_1}W = f(k)\,. \tag{60}$$

Since $u(k)$ and $f(k)$ are even, we consider equation (60) with $W_k(0) = 0$. It is easy to find a solution $u_1(k)$ of the corresponding homogeneous equations (60) for $k \geq 0$,

$$u_1(k) = k(k+\mu^{-1})e^{-\mu k}.$$

Then using reduction of order, we can obtain another linearly independent solution $u_2(k)$ with $u_2'(0) = 0$ and $u_1u_2' - u_1'u_2 = 1$. Also as $k \to +\infty$, $u_2(k)$ and its derivatives have order of $O(k^{-2}e^{\mu k})$. Thus the solution of (60) is

$$\begin{aligned} W(k) &= u_1(k)\int_0^k u_2(s)f(s)ds + u_2(k)\int_k^{+\infty} u_1(s)f(s)ds \\ &\stackrel{\text{def}}{=} \mathcal{N}(f(k)) \stackrel{\text{def}}{=} \int_0^{+\infty} \ker(k,s)f(s)ds \quad \text{for } k \geq 0\,, \end{aligned} \tag{61}$$

and evenly extended to $k < 0$.

Lemma 2. If $f(k) \in L^2_e(\mathbf{R})$, then $\mathcal{N}(f) \in H^2_e(\mathbf{R})$ and

$$\|\mathcal{N}(f)\|_{H^2_e(\mathbf{R})} \leq K\|f\|_{L^2_e(\mathbf{R})}\,.$$

Proof: Since $\mathcal{N}(f)$ is even,

$$\|\mathcal{N}(f)\|^2_{L^2_e(\mathbf{R})} = 4\|\mathcal{N}(f)\|^2_{L^2(\mathbf{R}^+)} \leq 4\left\| \int_0^{+\infty} \ker(k,s) f(s)ds \right\|^2_{L^2(\mathbf{R}^+)}$$
$$\leq 4\int_0^{+\infty} \left(\int_0^{+\infty} |\ker(k,s)|ds \right) \left(\int_0^{+\infty} |\ker(k,s)| f^2(s) ds \right) dk$$
$$\leq 4\left(\sup_{0\leq k<+\infty} \int_0^{+\infty} |\ker(k,s)|ds \right) \left(\sup_{0\leq s<+\infty} \int_0^{+\infty} |\ker(k,s)|dk \right) \|f\|_{L^2(\mathbf{R}^+)} .$$

But

$$\sup_{0\leq k<+\infty} \int_0^{+\infty} |\ker(k,s)|ds \leq \sup_{0\leq k<+\infty} \left(\int_0^{k} |u_1(k)u_2(s)|ds \right.$$
$$\left. + \int_k^{+\infty} |u_2(k)u_1(s)|ds \right) \leq K \sup_{0\leq k<+\infty} \left(\int_0^{k} (k^2+1)e^{-\mu k}(s^2+1)^{-1}e^{\mu s} ds \right.$$
$$\left. + \int_k^{+\infty} (k^2+1)^{-1}e^{\mu k}(s^2+1)e^{-\mu s} ds \right) \leq K_1 < +\infty .$$

The same estimate holds for $\sup_{0\leq s<+\infty} \int_0^{+\infty} |\ker(k,s)|dk$. Thus $\|\mathcal{N}f\|_{L^2_e(\mathbf{R})} \leq K\|f\|_{L^2_e(\mathbf{R})}$. By differentiating (61) once to have

$$\frac{d}{dk}\mathcal{N}(f(k)) = u_1'(k)\int_0^k u_2(s)f(s)ds + u_2'(k)\int_k^{+\infty} u_1(s)f(s)ds,$$

and using the same argument for $\mathcal{N}(f(k))$, we obtain $\|d(\mathcal{N}(f))/dk\|_{L^2_o(\mathbf{R})} \leq K\|f\|_{L^2_e(\mathbf{R})}$. Since $\mathcal{N}(f)$ satisfies (60), $\|d^2(\mathcal{N}(f))/dk^2\|_{L^2_e(\mathbf{R})} \leq K\|f\|_{L^2_e(\mathbf{R})}$. Therefore $\|\mathcal{N}f\|_{H^2_e(\mathbf{R})} \leq K\|f\|_{L^2_e(\mathbf{R})}$.

Thus the solution of (60) is $W(k) = \mathcal{N}(f)$ for $f \in L^2_e(\mathbf{R})$. By (57) and the definitions of T and $W(k)$, we can write (59) into

$$u(k) + \frac{3a\mu^2}{\rho|k|-\gamma_1}\mathcal{N}(f) = f,$$

or

$$u(k) = f(k) - \frac{3\mu^2 a}{\rho|k|-\gamma_1}\mathcal{N}(f). \tag{62}$$

From (62), Lemma 2, and the fact that the first order derivative of $|k|$ is piecewise continuous, we have that (59) has a unique solution $u \in L^2_e(\mathbf{R})$ with $\|u\|_{L^2(\mathbf{R})} = \|(I-T)^{-1}f\|_{L^2(\mathbf{R})} = \|W_{kk} - \mu^2 W\|_{L^2(\mathbf{R})} \leq K\|f\|_{L^2(\mathbf{R})}$ and

$$\|(I-T)^{-1}f\|_{H^1(\mathbf{R})} = \|u\|_{H^1(\mathbf{R})} \leq K\|f\|_{H^1(\mathbf{R})},$$
$$\|k(I-T)^{-1}f\|_{H^1(\mathbf{R})} = \|ku\|_{H^1(\mathbf{R})} \leq K(\|k(df/dk)\|_{L^2(\mathbf{R})} + \|f\|_{H^1(\mathbf{R})}). \tag{63}$$

Next let us consider

$$u - T_\epsilon(u) = f \tag{64}$$

for $\epsilon > 0$ small, where f and u are even in φ with $f \in L^2(\mathbf{R})$. We shall show that $I - T_\epsilon$ is invertible in $L^2_e(\mathbf{R})$ for $f \in L^2_e(\mathbf{R})$. Write (64) as

$$u - T(u) + (T - T_\epsilon)(u) = f. \tag{65}$$

Now we estimate the operator norm of $T - T_\epsilon$. If $u \in L^2_e(\mathbf{R})$, then by (56)

$$\begin{aligned}\mathcal{T}_\epsilon(u) \overset{\text{def}}{=} (T_\epsilon - T)(u) = &- \frac{3a\mu\rho|k|(\epsilon|k|\cosh\epsilon|k| - \sinh\epsilon|k|)}{2\rho\epsilon|k|\sinh\epsilon|k|(\rho|k| - \gamma_1)(\rho|k|(1+\rho\epsilon|k|k_2(k))^{-1} - \gamma_1)} \\ &\times \int_{-\infty}^{+\infty} e^{-\mu|k-k'|} u(k')dk'.\end{aligned}$$

Therefore

$$\begin{aligned}\|\mathcal{T}_\epsilon(u)\|^2_{L^2_e(\mathbf{R})} &\le K \int_{-\infty}^{+\infty} \frac{|k|^2(\epsilon|k|\cosh\epsilon|k| - \sinh\epsilon|k|)^2}{(\epsilon|k|\sinh\epsilon|k|)^2(\rho|k| - \gamma_1)^2(\rho|k|(1+\rho\epsilon|k|k_2(k))^{-1} - \gamma_1)^2} \\ &\quad \times \left(\int_{-\infty}^{+\infty} e^{-\mu|k-k'|} u(k')dk'\right)^2 dk \\ &\le K\|u\|^2_{L^2(\mathbf{R})} \int_{-\infty}^{+\infty} \frac{k^2(\epsilon|k|\cosh\epsilon|k| - \sinh\epsilon|k|)^2}{(|k|+1)^4(\epsilon|k|\sinh\epsilon|k|)^2} dk \\ &= K\|u\|^2_{L^2(\mathbf{R})} I^2(\epsilon).\end{aligned}$$

However

$$\left|\frac{\epsilon|k|\cosh\epsilon|k| - \sinh\epsilon|k|}{\epsilon|k|\sinh\epsilon|k|}\right| \le K_3 < +\infty \quad \text{for all } \epsilon > 0, k \in (-\infty, +\infty).$$

By the dominated convergence theorem, $I(\epsilon) \to 0$ and $\|\mathcal{T}_\epsilon\| \le KI(\epsilon) \to 0$ when $\epsilon \to 0$. For small $\epsilon > 0$ the equation of $u - T_\epsilon(u) = f$ can be transformed into

$$u - (I - T)^{-1}(T_\epsilon - T)u = (I - T)^{-1} f.$$

Since $(I-T)^{-1}$ is bounded from $L^2_e(\mathbf{R})$ to $L^2_e(\mathbf{R})$, the operator norm of $(I-T)^{-1}(T_\epsilon - T)$ goes to zero as $\epsilon \to 0$, which implies $I - (I-T)^{-1}(T_\epsilon - T)$ is invertible and the solution of $u - T_\epsilon(u) = f$ satisfies

$$\begin{aligned}\|u\|_{L^2_e(\mathbf{R})} &= \|(I - T_\epsilon)^{-1} f\|_{L^2_e(\mathbf{R})} \\ &\le \left\|\Big(I - (I-T)^{-1}(T_\epsilon - T)\Big)^{-1}(I-T)^{-1} f\right\|_{L^2_e(\mathbf{R})} \le K\|f\|_{L^2_e(\mathbf{R})},\end{aligned} \tag{66}$$

if $\epsilon > 0$ is small. By (64) and the definition of T_ϵ in (57), we also have that

$$\|ku\|_{L_o^2(\mathbf{R})} \leq \|kT_\epsilon(u)\|_{L_o^2(\mathbf{R})} + \|kf\|_{L_o^2(\mathbf{R})} \leq K(\|kf\|_{L_o^2(\mathbf{R})} + \|f\|_{L_e^2(\mathbf{R})}). \tag{67}$$

For $f(k) \in H^1(\mathbf{R})$, by taking the derivative of (64), we obtain

$$\begin{aligned} \left\|\frac{d}{dk}u(k)\right\|_{L^2(\mathbf{R})} + \left\|k\frac{d}{dk}u(k)\right\|_{L^2(\mathbf{R})} \leq & K\Big(\left\|k\frac{d}{dk}f(k)\right\|_{L^2(\mathbf{R})} + \left\|\frac{df(k)}{dk}\right\|_{L^2(\mathbf{R})} \\ & + \|(k^2+1)^{1/2}f(k)\|_{L^2(\mathbf{R})}\Big). \end{aligned} \tag{68}$$

If $(1+k^2)^{s/2}f(k)$ and $(1+k^2)^{s/2}(df/dk)$ are in $L^2(\mathbf{R})$, then by using (64) repeatedly and noting that the integral operator

$$\int_{-\infty}^{+\infty} e^{-\mu|k-k'|}(1+k^2)^{s/2}(1+|k'|^2)^{-s/2}f(k')dk'$$

maps $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$ continuously for any $s \geq 0$, we obtain

$$\begin{aligned} & \|(1+k^2)^{s/2}u(k)\|_{L^2(\mathbf{R})} + \left\|(1+k^2)^{s/2}\frac{d}{dk}u(k)\right\|_{L^2(\mathbf{R})} \\ & \qquad \leq K\left(\|(1+k^2)^{s/2}f(k)\|_{L^2(\mathbf{R})} + \left\|(1+k^2)^{s/2}\frac{d}{dk}f(k)\right\|_{L^2(\mathbf{R})}\right). \end{aligned} \tag{69}$$

In order to have (69) we have used the fact that $u = T_\epsilon(u) + f$ implies

$$\|ku\|_{L^2(\mathbf{R})} \leq \|kT_\epsilon(u)\|_{L^2(\mathbf{R})} + \|kf\|_{L^2(\mathbf{R})} \leq K(\|u\|_{L^2(\mathbf{R})} + \|kf\|_{L^2(\mathbf{R})}).$$

Now we can consider (58). From above derivation, for $\epsilon > 0$ small enough, (58) has a solution $\hat{F}[\omega]$ such that

$$\begin{aligned} & \|(1+k^2)^{s/2}\,\hat{F}[\omega]\|_{L_e^2(\mathbf{R})} + \left\|(1+k^2)^{s/2}\frac{d}{dk}\,\hat{F}[\omega]\right\|_{L_o^2(\mathbf{R})} \\ & \qquad \leq K\left(\|(1+k^2)^{s/2}g_0(k)\|_{L_e^2(\mathbf{R})} + \left\|(1+k^2)^{s/2}\frac{d}{dk}g_0(k)\right\|_{L_o^2(\mathbf{R})}\right), \end{aligned}$$

if the right hand side is finite. Then we use the Plancherel equality, the properties of the Fourier transform, and the definition of $g_0(k)$ in (55) to obtain that for $g(\varphi) \in W_e^{s-1}$ with $s \geq 1$, the equation (53) has a solution $\omega \stackrel{\text{def}}{=} N^{-1}(g) \in W_e^s$ satisfying

$$\|N^{-1}(g)\|_{W_e^s} \leq K\|g\|_{W_e^{s-1}}.$$

We summarize it as follows.

Theorem 3. If $N(\omega)$ is defined in (51) and $g(\varphi) \in W_e^s$ for $s \geq 0$, then for small $\epsilon > 0$ the equation $N(\omega) = g$ with $\omega_\varphi(0) = 0$ has a solution $\omega = N^{-1}(g)$ in W_e^{s+1} satisfying

$$\|N^{-1}(g)\|_{W_e^{s+1}} \leq K\|g\|_{W_e^s}\,.$$

6. Existence Proof

We consider (42), (43), (49), (50) and (52). Let

$$\begin{aligned}
&p(\varphi) = \omega(\varphi)\epsilon^{-1/2}, \quad p_1(\varphi) = \Big(\theta^+(\varphi,0) - L_1(S)\Big)\epsilon^{-1/2},\\
&p_2(\varphi) = \Big(\lambda^+(\varphi,0) - M_1(S)\Big)\epsilon^{-1/2},\\
&q_1(\varphi,\psi) = \xi(\varphi,\psi)\epsilon^{-1/2}, \quad q_2(\varphi,\psi) = \zeta(\varphi,\psi)\epsilon^{-1/2}.
\end{aligned} \tag{70}$$

Then we rewrite (42), (43), (49), (50) and (52) by

$$\begin{aligned}
p(\varphi) &= \epsilon^{-1/2}N^{-1}\Big(g_1\Big(\epsilon^{1/2}p, \epsilon^{1/2}q_1, \epsilon^{1/2}q_2, \epsilon^{1/2}p_2(\varphi) + M_1(S)\Big)\Big)\\
&= \tilde{P}(p, p_1, p_2, q_1, q_2),
\end{aligned} \tag{71}$$

$$\begin{aligned}
p_1(\varphi) &= \epsilon^{-1/2}\Big(L_2(p\epsilon^{1/2}, q_1\epsilon^{1/2}) + L_3\Big(p\epsilon^{1/2}, \epsilon^{1/2}q_1, \epsilon^{1/2}q_2, \epsilon^{1/2}p_2(\varphi) + M_1(S)\Big)\Big)\\
&= \tilde{P}_1(p, p_1, p_2, q_1, q_2),
\end{aligned} \tag{72}$$

$$\begin{aligned}
p_2(\varphi) &= \epsilon^{-1/2}\Big(M_2(p\epsilon^{1/2}, q_1\epsilon^{1/2}) + M_3\Big(p\epsilon^{1/2}, \epsilon^{1/2}q_1, \epsilon^{1/2}q_2, \epsilon^{1/2}p_2(\varphi) + M_1(S)\Big)\Big)\\
&= \tilde{P}_2(p, p_1, p_2, q_1, q_2),
\end{aligned} \tag{73}$$

$$\begin{aligned}
q_1(\varphi,\psi) &= \epsilon^{-1/2}\,\overset{\vee}{F}\Big[k_3(k,\psi)\,\hat{F}\Big[f_2\big((S+\epsilon^{1/2}p)_\varphi(\psi+1) + q_1\epsilon^{1/2},\\
&\qquad -(S+\epsilon^{1/2}p) + q_2\epsilon^{1/2}, M_1(S) + \epsilon^{1/2}p_2\big)\Big]\Big]\\
&= Q_1(p, p_1, p_2, q_1, q_2),
\end{aligned} \tag{74}$$

$$\begin{aligned}
q_2(\varphi,\psi) &= \epsilon^{-1/2}\,\overset{\vee}{F}\Big[(1/ik)k_{3\psi}(k,\psi)\,\hat{F}\Big[f_2\big((S+\epsilon^{1/2}p)_\varphi(\psi+1) + q_1\epsilon^{1/2},\\
&\qquad -(S+\epsilon^{1/2}p) + q_2\epsilon^{1/2}, M_1(S) + \epsilon^{1/2}p_2\big)\Big]\Big]\\
&= Q_2(p, p_1, p_2, q_1, q_2).
\end{aligned} \tag{75}$$

We are looking for fixed points of mapping $(\tilde{P}, \tilde{P}_1, \tilde{P}_2, Q_1, Q_2)$ in some Banach spaces. Define a closed convex set in a Banach space $W_e^{s+1} \times W_o^s \times W_e^s \times B_o^s \times B_e^s$ as follows:

$$\begin{aligned}\mathcal{S} =\Big\{ & X = (p, p_1, p_2, q_1, q_2) \in W_e^{s+1} \times W_o^s \times W_e^s \times B_o^s \times B_e^s \ \Big| \\ & |\!|\!| X |\!|\!| = \|p\|_{W_e^{s+1}} + \|p_1\|_{W_o^s} + \|p_2\|_{W_e^s} + \|q_1\|_{B_o^s} + \|q_2\|_{B_e^s} \le b_1 < +\infty \Big\},\end{aligned}$$

where $b_1 > 0$ is a small fixed number with $s \ge 2$. Assume that $X = (p, p_1, p_2, q_1, q_2) \in \mathcal{S}$. By Theorem 2, we obtain

$$\|Q_1(p, p_1, p_2, q_1, q_2)\|_{B_o^s} + \|Q_2(p, p_1, p_2, q_1, q_2)\|_{B_e^s} \le K\epsilon^{1/2}. \tag{76}$$

In order to use Theorems 1 and 3, we need to change (71) to (73) into the following forms,

$$p(\varphi) = \tilde{P}(p, p_1, p_2, Q_1, Q_2) = P(p, p_1, p_2, q_1, q_2), \tag{77}$$

$$p_1(\varphi) = \tilde{P}_1(P, p_1, p_2, Q_1, Q_2) = P_1(p, p_1, p_2, q_1, q_2), \tag{78}$$

$$p_2(\varphi) = \tilde{P}_2(P, p_1, p_2, Q_1, Q_2) = P_2(p, p_1, p_2, q_1, q_2). \tag{79}$$

From Theorem 3, we need to show that

$$\left\| g_1\big(\epsilon^{1/2}p, \epsilon^{1/2}Q_1, \epsilon^{1/2}Q_2, \epsilon^{1/2}p_2 + M_1(S)\big) \right\|_{W_e^s} \le K\epsilon. \tag{80}$$

By (48) and noting that

$$\begin{aligned}|I_1| &\stackrel{\text{def}}{=} \left| \overset{\vee}{F} \left[|k|\rho\epsilon|k|k_2(k)(1+\rho\epsilon|k|k_2(k))^{-1} \hat{F} [S(\varphi)] \right] \right| \\ &\le K \left| \overset{\vee}{F} \left[|k|\rho\epsilon|k|k_2(k)(1+\rho\epsilon|k|k_2(k))^{-1} \exp(-\mu|k|) \right] \right|.\end{aligned}$$

Since $|k|^2 e^{-\mu|k|}$ decays exponentially as $k \to +\infty$ and $k_2(k)(1+\rho\epsilon|k|k_2(k))^{-1}$ is uniformly bounded, $\|I_1\|_{W_e^s} \le K\epsilon$ for any $s \ge 0$. Therefore the W_e^s-norm of first two terms in g_1 of (48) are less than $K\epsilon$. Next we need to show $\int_\varphi^{+\infty} \tilde{f}_2(S+\omega, \xi, \zeta, \lambda^+)d\varphi \in W_e^s$. By the definition of $\tilde{f}_2$ in (44),

$$\begin{aligned}\tilde{f}_2(S+\omega, \xi, \zeta, \lambda^+) &= \gamma_1\xi + [-3(\zeta(S+\omega)_\varphi - \xi(S+\omega) + \zeta\xi) \\ &\quad + \epsilon\mathcal{R}((S+\omega)_\varphi + \xi, -(S+\omega) + \zeta, \lambda^+)] \\ &\stackrel{\text{def}}{=} \gamma_1\xi + II_1(\omega, \xi, \zeta).\end{aligned} \tag{81}$$

Now we need

Lemma 3. If $u_1(\varphi) \in W_o^s, u_2(\varphi) \in W_e^s$ for $s \geq 1$, then $\int_\varphi^{+\infty} u_1(\varphi)u_2(\varphi)d\varphi \in W_e^{s+1}$ and

$$\Big\| \int_\varphi^{+\infty} u_1(\varphi)u_2(\varphi)d\varphi \Big\|_{W_e^{s+1}} \leq K\|u_1(\varphi)\|_{W_o^s}\|u_2(\varphi)\|_{W_e^s}\,.$$

Proof: Let $v(\varphi) = \int_\varphi^{+\infty} u_1(\varphi)u_2(\varphi)d\varphi$. Then $(dv/d\varphi) = u_1u_2 \in W_o^s$ and by Lemma 1

$$\|(dv/d\varphi)\|_{W_o^s} \leq K\|u_1(\varphi)\|_{W_o^s}\|u_2(\varphi)\|_{W_e^s}\,.$$

Thus we only need to show the estimates for $\|v\|_{W_e^0}$. Since v is even,

$$\begin{aligned}
\|v\|^2_{L^2_e(\mathbf{R})} &= 4\int_0^{+\infty}\Big|\int_\varphi^{+\infty} u_1(t)u_2(t)dt\Big|^2 d\varphi \\
&\leq K\int_0^{+\infty}(\varphi^2+1)^{-2}\Big|\int_\varphi^{+\infty} u_1(t)u_2(t)(1+t^2)dt\Big|^2 d\varphi \\
&\leq K\|(1+\varphi^2)^{1/2}u_1\|^2_{L^2(\mathbf{R})}\|(1+\varphi^2)^{1/2}u_2\|^2_{L^2(\mathbf{R})}\,,
\end{aligned}$$

and

$$\begin{aligned}
\|\varphi v\|^2_{L^2_o(\mathbf{R})} &\leq K\int_0^{+\infty}\varphi^2(\varphi^2+1)^{-2}\Big|\int_\varphi^{+\infty} u_1(t)u_2(t)(1+t^2)dt\Big|^2 d\varphi \\
&\leq K\|(1+\varphi^2)^{1/2}u_1\|^2_{L^2(\mathbf{R})}\|(1+\varphi^2)^{1/2}u_2\|^2_{L^2(\mathbf{R})}\,.
\end{aligned}$$

Thus $\|v\|_{W_e^0} \leq K\|u_1\|_{W_o^s}\|u_2\|_{W_e^s}$ and the proof is completed.

From (81), it is easy to check that every term in II_1 is either a product of u_1 and u_2 with $u_1 \in W_e^{s-1}, u_2 \in W_o^{s-1}$ or a term like $\overset{\vee}{F}\,[|k|\,\hat{F}\,[f(\varphi)]]$ with $\|f\|_{W_o^s} \leq K\epsilon$. The integral of the latter is $\overset{\vee}{F}\,[(|k|/ik)\,\hat{F}\,[f(\varphi)]] \in W_e^s$. Thus

$$\Big\| \int_\varphi^{+\infty} II_1 d\varphi \Big\|_{W_e^s} \leq K(\epsilon + \|\zeta\|_{W_e^s} + \|\xi\|_{W_o^s})\,.$$

But by the definition of P in (77),

$$\begin{aligned}
\tilde{f}_2(S+\omega, \epsilon^{1/2}Q_1, \epsilon^{1/2}Q_2, \lambda^+) &= \gamma_1 Q_1|_{\psi=0} + II_1(\omega, \epsilon^{1/2}Q_1, \epsilon^{1/2}Q_2, \lambda^+) \\
&= II_1(S+\omega, \epsilon^{1/2}Q_1, \epsilon^{1/2}Q_2, \lambda^+)\,.
\end{aligned}$$

since $Q_1|_{\psi=0} = 0$ by our definition of Q_1 in (74) and the construction of $k_3(k, \psi)$ in (36). Thus

$$\left\| \int_{\varphi}^{+\infty} \tilde{f}(S+\omega, \epsilon^{1/2}Q_1, \epsilon^{1/2}Q_2, \lambda^+)d\varphi \right\|_{W_e^s}$$
$$\leq K(\epsilon + \|\epsilon^{1/2}Q_1\|_{W_o^s} + \|\epsilon^{1/2}Q_1\|_{W_e^s}) \leq K\epsilon\,,$$

and (80) is proved. From Theorem 3,

$$\|P(p, p_1, p_2, q_1, q_2)\|_{W_e^{s+1}} \leq K\epsilon^{1/2}\,. \tag{82}$$

Finally by (78), (79) and Theorem 1,

$$\|P_1(p, p_1, p_2, q_1, q_2)\|_{W_o^s} + \|P_2(p, p_1, p_2, q_1, q_2)\|_{W_e^s} \leq K\epsilon^{1/2}\,. \tag{83}$$

Define

$$\mathcal{T}(p, p_1, p_2, q_1, q_2) = [P, P_1, P_2, Q_1, Q_2](p, p_1, p_2, q_1, q_2).$$

By (76), (82) and (83), $\mathcal{T}$ maps $\mathcal{S}$ into itself if $\epsilon > 0$ is small. Also by using the same proof as (76), (82) and (83) from Theorems 1, 2 and 3, we have

Theorem 4. If $X^{(1)} = (p^{(1)}, p_1^{(1)}, p_2^{(1)}, q_1^{(1)}, q_2^{(1)})$ and $X^{(2)} = (p^{(2)}, p_1^{(2)}, p_2^{(2)}, q_1^{(2)}, q_2^{(2)})$ are in $\mathcal{S}$, then

$$\|\!|\mathcal{T}(X^{(1)}) - \mathcal{T}(X^{(2)})|\!\| \leq K\epsilon^{1/2}\|\!|X^{(1)} - X^{(2)}|\!\|\,.$$

We choose ϵ so small such that the constant $K\epsilon^{1/2}$ in Theorem 4 is also less than 1/2. Then $\mathcal{T}$ is a contraction in $\mathcal{S}$. By the contraction mapping theorem, we can have a unique fixed point of $\mathcal{T}$ in $\mathcal{S}$. Therefore we obtain a solution (p, p_1, p_2, q_1, q_2) in $\mathcal{S}$ for (71) to (75). However from (73), (79) and (50), it is easy to see that if $\|\omega_\varphi\|_{W_o^s} + \|\xi(\varphi, 0)\|_{W_o^s} \leq K\epsilon$ and $\|\lambda^+\|_{W_e^s} \leq K$, then $\|P_2(p, p_1, p_2, q_1, q_2)\|_{W_e^s} \leq K\epsilon^{3/2}$. Therefore we have the following Theorem.

Theorem 5. For $\epsilon > 0$ small enough, the integral equations (32) to (35) have a solution

$$\theta^+(\varphi, \psi) = \overset{\vee}{F}\left[e^{-\epsilon|k|\psi}\,\hat{F}[S_\varphi]\right] + \theta_1^+(\varphi, \psi)\,,$$

$$\lambda^+(\varphi, \psi) = \overset{\vee}{F}\left[(i|k|\epsilon/k)e^{-\epsilon|k|\psi}\,\hat{F}[S_\varphi]\right] + \lambda_1^+(\varphi, \psi)\,,$$

$$\theta^-(\varphi, \psi) = (\psi + 1)S_\varphi + \theta_1^-(\varphi, \psi)\,,$$

$$\lambda^-(\varphi, \psi) = -S + \lambda_1^-(\varphi, \psi)\,,$$

where

$$S(\varphi) = \frac{a\mu^2}{\varphi^2 + \mu^2}$$

with $a = -(4\gamma_1/3), \mu = -(\rho/\gamma_1)$, $\theta_1^{\pm}$ are odd and $\lambda_1^{\pm}$ are even in φ, $(1+\varphi^2)^{1/2}\theta_1^{\pm}(\cdot,\psi), (1+\varphi^2)^{1/2}\lambda_1^{\pm}(\cdot,\psi) \in H^s(\mathbf{R})$ for $s \geq 0$ with

$$\begin{aligned}
&\sup_{-1\leq\psi\leq 0} \left(\|(1+\varphi^2)^{1/2}\theta_1^-(\cdot,\psi)\|_{H^s(\mathbf{R})} + \|(1+\varphi^2)^{1/2}\lambda_1^-(\cdot,\psi)\|_{H^s(\mathbf{R})} \right) \\
&\quad + \sup_{0\leq\psi<\infty} \left(\|\theta_1^+(\cdot,\psi)\|_{H^s(\mathbf{R})} + \|\lambda_1^+(\cdot,\psi)\|_{H^s(\mathbf{R})} \right) \\
&\quad + \sup_{0\leq\psi\leq K_1} \left(\|\varphi\theta_1^+(\cdot,\psi)\|_{H^s(\mathbf{R})} + \|\varphi\lambda_1^+(\cdot,\psi)\|_{H^s(\mathbf{R})} \right) \\
&\leq K\epsilon,
\end{aligned}$$

and K is a constant independent of ϵ but may depend on s and K_1.

Since $H^1(\mathbf{R})$ is embedded in bounded continuous function space $C^0(\mathbf{R})$ and $\theta_1^{\pm}, \lambda_1^{\pm}$ satisfy the Cauchy-Riemann equations (20) or (24), $(1+\varphi^2)^{1/2}\theta_1^{\pm}, (1+\varphi^2)^{1/2}\lambda_1^{\pm}$ and their derivatives are bounded and continuous in either $\mathbf{R}\times(-1,0)$ or $\mathbf{R}\times(0,+\infty)$. Then from (34), (35) and (40), we write

$$f_2(\theta^-, \lambda^-, \lambda^+) = \gamma_1\epsilon\theta^- + \mathcal{R}_1\,,$$

where $\mathcal{R}_1$ consists of all nonlinear terms in f_2 and $\varphi^2\mathcal{R}_1 \in H^s(\mathbf{R})$ for $s \geq 0$. Therefore (42) can be written as

$$\xi = \overset{\vee}{F}\left[k_3(k,\psi)\,\hat{F}\,[\gamma_1\epsilon(\eta_\varphi + \xi)]\right] + \overset{\vee}{F}\left[k_3(k,\psi)\,\hat{F}\,[\mathcal{R}_1)]\right]. \tag{84}$$

But from (45), we have

$$\hat{F}\,[\epsilon\eta_\varphi] = \left(-\gamma_1 + \rho|k|(1+\rho\epsilon|k|k_2(k))^{-1}\right)^{-1}\left(\hat{F}\,[\gamma_1\epsilon\xi] + \hat{F}\,[\mathcal{R}_1]\right).$$

Substituting this expression into (84) and taking the Fourier and inverse-Fourier transforms, we can obtain

$$\begin{aligned}
\xi(\varphi,\psi) &= \overset{\vee}{F}\left[\frac{k_3(k,\psi)(1+\gamma_1(-\gamma_1+\rho|k|(1+\rho\epsilon|k|k_2(k))^{-1})^{-1})\,\hat{F}\,[\mathcal{R}_1]}{1-\gamma_1\epsilon k_3(k,\psi)(1+\gamma_1(-\gamma_1+\rho|k|(1+\rho\epsilon|k|k_2(k))^{-1})^{-1})}\right] \\
&\overset{\text{def}}{=} \overset{\vee}{F}\left[k_4(k,\psi)\,\hat{F}\,[\mathcal{R}_1]\right].
\end{aligned}$$

However, notice that $\gamma_1 < 0$ and $k_4(k, \psi)$ is bounded and even in k. Since $\mathcal{R}_1$ is odd and $\varphi^2\mathcal{R}_1$ is in $H^s(\mathbf{R})$ for $s \geq 0$, integration by parts twice yields

$$\varphi^2\xi(\varphi, \psi) = \int_0^{+\infty} \sin(k\varphi)\left(\frac{d^2}{dk^2}(k_4(k,\psi)\hat{F}[\mathcal{R}_1])\right)dk\,.$$

But $\varphi^2\mathcal{R}_1$ is in $H^s(\mathbf{R})$ and the derivatives of $k_4(k,\psi)$ are bounded, which implies $\varphi^2\xi \in H^s(\mathbf{R})$ for $s \geq 0$ and any fixed ψ. Then by (45), $\varphi^2\eta_\varphi \in H^s(\mathbf{R})$. Therefore from (40) $\varphi^2\theta^- \in H^s(\mathbf{R})$ for $s \geq 0$. Now we use integration by parts in (43) and the oddness of $(1/ik)k_{3\psi}(k,\psi)$ to have $(1+\varphi^2)\zeta \in C^s(\mathbf{R})$ for integer $s \geq 0$. From (45) and the fact that $\xi_{\psi=0} = Q_1|_{\psi=0} = 0$ in (74),

$$\eta = \overset{\vee}{F}\left[\left(-\gamma_1 + \rho|k|(1+\rho\epsilon|k|k_2(k))^{-1}\right)^{-1}\hat{F}\left[\int_\varphi^{+\infty}\mathcal{R}_2 d\varphi\right]\right],$$

where $\mathcal{R}_2 = (3/2)(\eta^2)_\varphi + \tilde{f}_2$, and $\varphi^3\mathcal{R}_2, \varphi^2\int_\varphi^{+\infty}\mathcal{R}_2 d\varphi \in H^s(\mathbf{R})$ using Lemma 3. Thus integration by parts twice yields $(1+\varphi^2)\eta \in C^s(\mathbf{R})$ for $s \geq 0$. Finally we use the Fourier transforms of $\exp(-|k|\epsilon\psi)$ and $\epsilon i(|k|/k)\exp(-\epsilon|k|\psi)$ in (49) and (50) and the classical estimates of elliptic operators in [13] to obtain

$$\sup_{\varphi\in\mathbf{R},\psi\in\mathbf{R}+}\left((|\theta^+| + |\lambda^+|)(1+\varphi^2+|\psi\epsilon|)\right) \leq K$$

and the derivatives of θ^+, λ^+ are bounded by $K(1+\varphi^2+|\psi\epsilon|^2)^{-1}$. Therefore Theorem 5 can be improved as follows.

Theorem 6. (32) to (35) have a solution stated in Theorem 5 with $\theta^\pm$ and $\lambda^\pm$ satisfying

$$\sup_{\varphi,\psi}\left((|\theta_1^\pm(\varphi,\psi)| + |\lambda_1^\pm(\varphi,\psi)|)(1+\varphi^2+|\epsilon\psi|)\right)$$
$$+\sum_{i=1}^{s}\sup_{\varphi,\psi}\left((|D^i\theta_1^\pm(\varphi,\psi)| + |D^i\lambda_1^\pm(\varphi,\psi)|)(1+\varphi^2+|\epsilon\psi|^2)\right) \leq K\epsilon$$

for any integer $s \geq 0$, where D^i is denoted as the i-th derivative with respect to either φ or ψ or both and K is independent of ϵ but may depend on s.

After transforming back to $\hat{\theta}^\pm, \hat{\lambda}^\pm$, and φ_1, ψ_1 in (30) and (31), and calculating the Fourier transforms explicitly, we have

Corollary 1. For $\epsilon > 0$ small, the equations (20), (21), (24) to (26) have a solution

$$\hat{\theta}^+(\varphi_1, \psi_1) = \frac{-2\epsilon^3 \varphi_1 a\mu(\epsilon\psi_1 + \mu)}{((\epsilon\psi_1 + \mu)^2 + (\epsilon\varphi_1)^2)^2} + \epsilon^2 \theta_1^+(\epsilon\varphi_1, \psi_1),$$
$$\hat{\lambda}^+(\varphi_1, \psi_1) = \frac{\epsilon^2 a\mu((\epsilon\psi_1 + \mu)^2 - (\epsilon\varphi_1)^2)}{((\epsilon\psi_1 + \mu)^2 + (\epsilon\varphi_1)^2)^2} + \epsilon^2 \lambda_1^+(\epsilon\varphi_1, \psi_1),$$
$$\hat{\theta}^-(\varphi_1, \psi_1) = \frac{-2\epsilon^3 \varphi_1 a\mu^2(\psi_1 + 1)}{(\mu^2 + (\epsilon\varphi_1)^2)^2} + \epsilon^2 \theta_1^-(\epsilon\varphi_1, \psi_1),$$
$$\hat{\lambda}^-(\varphi_1, \psi_1) = -\frac{\epsilon a\mu^2}{\mu^2 + (\epsilon\varphi_1)^2} + \epsilon\lambda_1^-(\epsilon\varphi_1, \psi_1),$$

where $a = -(4\gamma_1/3), \mu = -(\rho/\gamma_1)$, $\theta_1^\pm$ and $\lambda_1^\pm$ satisfy the properties stated in Theorem 6.

From the Bernoulli's equation (6) at the interface $\psi_1 = 0$, the interface of two fluids is determined by

$$\begin{aligned} y &= (1 + \gamma_1\epsilon)^{-1}(1/2)((1 - e^{2\hat{\lambda}^-}) - \rho(1 - e^{2\hat{\lambda}^+}))\Big|_{\psi_1 = 0} \\ &= -\hat{\lambda}^- + \epsilon Y_1(\epsilon\varphi_1) = \epsilon S(\epsilon\varphi_1) + \epsilon Y(\epsilon\varphi_1), \end{aligned}$$

where $\sum_{i=0}^{s} \sup_{\varphi\in\mathbf{R}} |(1 + \varphi^2)(d^s Y(\varphi)/d\varphi^s)| \leq K\epsilon$ and K is independent of ϵ. Finally we use the dimensional variables to state the existence of the solitary waves.

Theorem 7. There is a solitary wave moving with constant speed

$$U = \left((\rho^- - \rho^+)gh/(\rho^-(1 + \gamma_1\epsilon))\right)^{1/2}$$

in a two-fluid flow with $\gamma_1 < 0$ fixed and $\epsilon > 0$ a small parameter, where the height of the upper fluid is infinite while the depth of the lower fluid is h, and the densities of the upper and lower fluids are ρ^+ and ρ^-, respectively. The profile of the interface between two fluids is given by

$$y^* = \eta^*(x^*) = \frac{\epsilon a\mu^2 h^3}{(\epsilon x^*)^2 + (h\mu)^2} + \epsilon Y^*(\epsilon x^*/h),$$

where $a = -(4\gamma_1/3), \mu = -(\rho^+/\rho^-\gamma_1)$,

$$\sum_{i=0}^{s} \sup_{x\in\mathbf{R}} |(1 + x^2)(d^s Y^*(x)/dx^s)| \leq K\epsilon,$$

and K is independent of ϵ.

Obviously the first-order approximation of the solution given in Theorem 7 was obtained by Benjamin [4] using asymptotic method.

References

[1] C. J. Amick, On the theory of internal waves of permanent form in fluids of great depth, Trans. AMS, in press.

[2] C. J. Amick and J. F. Toland, Uniqueness and related analytic properties for the Benjamin-Ono equation – a nonlinear Neumann problem in the plane, Acta Math. 167 (1991), 107-126.

[3] C. J. Amick and J. F. Toland, Uniqueness of Benjamin's solitary wave solution of the Benjamin-Ono equation, IMA J. Appl. Math. 46 (1991), 21-28.

[4] T. B. Benjamin, Internal waves of permanent form in fluids of great depth, J. Fluid Mech. 29 (1967), 559-592.

[5] T. B. Benjamin, The stability of solitary waves, Proc. R. Soc. Lond. A 328 (1972), 153-183.

[6] T. B. Benjamin, A new kind of solitary wave, J. Fluid Mech. 245 (1992), 401-411.

[7] T. B. Benjamin, J. L. Bona and D. K. Bose, Solitary-wave solutions of nonlinear problems, Phil. Trans. R. Soc. Lond. A 331 (1990), 195-244.

[8] D. Bennett, J. L. Bona, R. Brown, S. Stansfield and J. Stroughair, The stability of internal waves, Proc. Camb. Phil. Soc. 94 (1983), 351-379.

[9] T. L. Bock and M. D. Kruskal, A two-parameter Miura transformation for the Benjamin-Ono equation, Phys. Lett. 74A (1980), 173-176.

[10] J. L. Bona, On the stability of solitary waves, Proc. R. Soc. Lond. A 344 (1975), 363-374.

[11] R. E. Davis and A. Acrivos, Solitary internal waves in deep water, J. Fluid Mech. 29 (1967), 593-607.

[12] A. S. Fokas and M. J. Ablowitz, The inverse scattering transform for the Benjamin-Ono equation – a pivot to multidimensional equations, Stud. Appl. Math. 68 (1983), 1-10.

[13] G. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, 2nd ed., Springer-Verlag, 1983.

[14] T. Levi-Civita, Dètermination rigoureuse des ondes permanentes d'ampleur finie, Mathematische Annalen 93 (1925), 264-314.

[15] K. Kirchgässner, private communication.

[16] S. A. Maslowe and L. G. Redekopp, Long nonlinear waves in stratified shear flows, J. Fluid Mech. 101 (1980), 321-348.

[17] Y. Matsuno, Exact multi-soliton solution of the Benjamin-Ono equation, J. Phys. A: Math. Gen. 12 (1979), 619-621.

[18] T. Maxworthy, On the formation of nonlinear internal waves from the gravitational collapse of mixed regions in two and three dimensions, J. Fluid Mech. 96 (1980), 47-64.

[19] H. Ono, Algebraic solitary waves in stratified fluids, J. Phys. Soc. Japan 39 (1975), 1082-1091.

[20] J. J. Stoker, Water Waves, Interscience Publishers, Inc., New York, 1957.

[21] E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals, 2nd ed., Oxford Univ. Press, London, 1975.

[22] K. Yosida, Functional Analysis, Springer-Verlag, New York, 1980.